Back paper exam Real Analysis I Time: 3hrs Total Marks: 50
Answer all and each question is worth 10 Marks Total Marks 50

1. Let (a_n) and (b_n) be two convergent sequences. Then prove that

   (i) $\lim a_n + \lim b_n = \lim(a_n + b_n)$;

   (ii) $\lim ca_n = c \lim a_n$ for any $c \in \mathbb{R}$;

   (iii) $\lim a_n - \lim b_n = \lim(a_n - b_n)$ and

   (iv) $\lim a_n \lim b_n = \lim a_n b_n$.

2. (a) Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function and $x \in \mathbb{R}$. Prove that $f(x_n) \to f(x)$ for any sequence (x_n) converging to x.

   (c) Let $f: [a, b] \to \mathbb{R}$ be a continuous one-one function. Prove the f is strictly monotone.

3. (a) Let $f: [a, b] \to \mathbb{R}$ be a continuous function. Prove that there exists $x, y \in [a, b]$ such that $f(x) \leq f(t) \leq f(y)$ for all $t \in [a, b]$.

   (b) Let $\emptyset \neq C \subset \mathbb{R}$ and define $f: \mathbb{R} \to \mathbb{R}$ by $f(t) = \inf_{x \in C} |x - t|$ for $t \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$.

4. (a) Suppose $f: \mathbb{R} \to \mathbb{R}$ is a differentiable function with derivative f' and f' is continuous. Suppose $f'(a) > 0$ for some $a \in \mathbb{R}$. Then there is a $\delta > 0$ such that f is strictly increasing in $(a - \delta, a + \delta)$.

   (b) Suppose $f: [a, b] \to \mathbb{R}$ be a differentiable function. Prove that f' has IVP.

5. (a) Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function with derivative f'. Suppose there is a $c \in (0, 1)$ such that $|f'(x)| < c$ for all $x \in \mathbb{R}$. Prove that there is a unique x_0 such that $f(x_0) = x_0$ and $f^n(x) \to x_0$ as $n \to \infty$ for all $x \in \mathbb{R}$.

   (b) Let $f: [a, b] \to [c, d]$ be a bijection and $x_0 \in [a, b]$. Suppose f is differentiable at x_0, f^{-1} is continuous at $y_0 = f(x_0)$ and $f'(x_0) \neq 0$. Prove that f^{-1} is differentiable at y_0 and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$.